Financial Modeling, 4th edition
Chapter 26: Normal versus Lognormal (Stock returns vs Stock prices)

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Reminder

It's completely unfair to say that this was discussed in your introductory finance course!

\[ P_0 \times \text{Exp}(\text{Ln}(P_T/P_0)) = P_T \]

Because \( \text{Exp}(\text{Ln}(x)) = x \)

Discrete returns

\[ P_0 \times (1 + \text{return}) = P_T \]

Continuous returns

\[ P_0 \times (1 + \text{return}) = P_T \]

Stock returns are normal

\[ \mu \Delta t + \sigma \sqrt{\Delta t} \cdot Z \]

where

\[ \mu = \text{mean return} \]

\[ \sigma = \text{standard deviation of return} \]

\[ Z = \text{random number, normally distributed, } \mu = 0, \sigma = 1. \text{ Simulated with Excel function Norm.S.Inv(rand()))} \]

Checking the simulation

Use Excel's Frequency to see the distribution

![Stock return simulation](image)

![Frequency distribution of returns](image)

PV AND FUTURE VALUE ARE INVERSES

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_0 )</td>
<td>( P_T )</td>
<td></td>
</tr>
<tr>
<td>Continuous returns</td>
<td>Continuous future value</td>
<td></td>
</tr>
<tr>
<td>( P_0 \times \text{Exp}(\text{Ln}(P_T/P_0)) )</td>
<td>( P_T )</td>
<td></td>
</tr>
<tr>
<td>( P_0 \times (1 + \text{return}) = P_T )</td>
<td>( P_T )</td>
<td></td>
</tr>
<tr>
<td>( P_0 \times \text{Exp}(\text{Ln}(P_T/P_0)) )</td>
<td>( P_T )</td>
<td></td>
</tr>
</tbody>
</table>

STOCK RETURNS ARE NORMAL

1,000 simulations

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>( \sigma )</td>
<td>( \Delta t )</td>
<td>( \text{Simulated} )</td>
</tr>
</tbody>
</table>
| 11% | 15% | 0.083333 | \( \text{Norm.S.Inv} \text{Rand())} \)

Annualizing return statistics:

<table>
<thead>
<tr>
<th>Annualizing return statistics:</th>
<th>Multiply by ( 1/\Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count</td>
<td>( \text{COUNT(A:A)} )</td>
</tr>
<tr>
<td>Mean</td>
<td>( \text{AVERAGE(A:A)} )</td>
</tr>
<tr>
<td>StdDev</td>
<td>( \text{STDEV.S(A:A)} )</td>
</tr>
<tr>
<td>Min</td>
<td>( \text{MIN(A:A)} )</td>
</tr>
<tr>
<td>Max</td>
<td>( \text{MAX(A:A)} )</td>
</tr>
</tbody>
</table>

Frequency distribution of returns

Annual (mean, sigma) = (11%, 15%), \( \Delta t = 0.004 \)
1+stock returns are lognormal

\[ \exp(\mu \Delta t + \sigma \sqrt{\Delta t} \cdot Z) \]

where

\( \mu = \text{mean return} \)

\( \sigma = \text{standard deviation of return} \)

Simulate 1+return: the growth of the stock price

1+return shows familiar “zigzag” noise

Frequency distribution of 1+return is lognormal

Lognormal harder to see when \( \Delta t \) is small

Frequency distribution of 1+return = \( \exp(\mu \Delta t + \sigma \sqrt{\Delta t} \cdot Z) \)

Annual (mean,sigma) = (12%,33%), \( \Delta t = 1.000 \)

1,000 simulations with \( \Delta t = 1 \) means 1,000 years of simulated stock growth!

Lognormal distribution, \( \mu = 0, \sigma = \{1, 0.5, 0.25\} \)

1,000 simulations with \( \Delta t = 1/250 \) means 4 years of simulated stock growth. Much harder to see the difference between normal and lognormal.
Normal approximation to the Lognormal distribution

When the Lognormal distribution $\log(\mu, \sigma)$ has an arithmetic mean $\mu$ that is much larger than its arithmetic standard deviation $\sigma$, the distribution tends to look like a Normal($\mu, \sigma$).

Lognormal $\log(\mu, \sigma)$ = Normal($\mu, \sigma$).

A generic use of limits to this approximation is $\mu > 8\sigma$. This approximation is not really useful from the point of view of simplifying the mathematics. But it is helpful in being able to quickly think of the range of the distribution and its peak in such circumstances. For example, we know that 55.7% of a Normal distribution is contained within a range $\pm 3\sigma$ from the mean $\mu$. So for a Lognormal(10, 2), we would estimate the distribution to be almost completely contained within a range $\pm 3\sigma$ and that it peaks at a little below 10 (remember that the mode, median and mean appear in that order from left to right for a right skewed distribution).

© Vose Software 2006. Reference Number: M-59227A. When the mean is large relative to the sigma, the lognormal looks like the normal.

Source: http://www.vosesoftware.com/vosesoftware/ModelRiskHelp/index.htm#Distributions/Approximating_one_distribution_with_another/Normal_approximation_to_the_Lognormal_distribution.htm

Theorem: Sum of normals is normal

Independent random variables \( X \) and \( Y \) are independent random variables that are normally distributed (and therefore also jointly so), then their sum is also normally distributed, i.e., if

\[ X \sim N(\mu_X, \sigma_X^2) \]
\[ Y \sim N(\mu_Y, \sigma_Y^2) \]

then

\[ Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2) \]

This means that the sum of two independently distributed normal random variables is normal, with its mean being the sum of the two means, and its variance being the sum of the two varians (i.e., the square of the standard deviation is the sum of the squares of the standard deviations).

Note that the result that the sum is normally distributed requires the assumption of independence; not just uncorrelatedness, two (or more) jointly normally distributed random variables can be uncorrelated without being independent, in which case their sum can be non normally distributed (see Normally distributed and uncorrelated does not imply independently symmetric, example). The result also holds in all cases, while the result for the variance requires uncorrelatedness, but not independence.

Multiplying continuous returns means adding them inside the Exp[] function

\[
\text{Exp} \left[ \mu \Delta t + \sigma \sqrt{\Delta t} * Z_1 \right] * \text{Exp} \left[ \mu \Delta t + \sigma \sqrt{\Delta t} * Z_2 \right] * ... * \text{Exp} \left[ \mu \Delta t + \sigma \sqrt{\Delta t} * Z_N \right] \\
= \text{Exp} \left[ \mu \Delta t + \mu \Delta t + ... + \mu \Delta t + \sigma \sqrt{\mu \Delta t} * (Z_1 + Z_2 + ... + Z_N) \right] \\
= \text{Exp} \left[ \mu (n \Delta t) + \sigma \sqrt{n \Delta t} \right]
\]

The first two moments of the terminal stock price when the return is normally distributed

\[ E[S_T] = S_0 \exp \left[ \left( \mu + \frac{\sigma^2}{2} \right) T \right] \]
\[ \sigma = S_0 \exp \left[ \left( \mu + \frac{\sigma^2}{2} \right) T \right] \sqrt{T \left[ \exp \left( \sigma^2 - 1 \right) \right]} \]

Theorem: $1+growth$ is additive

\[
\text{Exp} \left[ \mu \Delta t + \sigma \sqrt{\Delta t} * Z_1 \right] * \text{Exp} \left[ \mu \Delta t + \sigma \sqrt{\Delta t} * Z_2 \right] * ... * \text{Exp} \left[ \mu \Delta t + \sigma \sqrt{\Delta t} * Z_N \right] \\
= \text{Exp} \left[ \mu \Delta t + \mu \Delta t + ... + \mu \Delta t + \sigma \sqrt{\mu \Delta t} * (Z_1 + Z_2 + ... + Z_N) \right] \\
= \text{Exp} \left[ \mu (n \Delta t) + \sigma \sqrt{n \Delta t} \right]
\]

The first two moments of the terminal stock price when the return is normally distributed

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\[ \sigma = S_0 \exp \left[ \left( \mu + \frac{\sigma^2}{2} \right) T \right] \sqrt{T \left[ \exp \left( \sigma^2 - 1 \right) \right]} \]
What does this mean? (1)

- The expected stock price increases with time and variance:
  \[ E[S_T] = S_0 \exp \left( \mu + \frac{\sigma^2}{2} \right) T \]

- This is why in risk-neutral pricing, the mean is often written as \( \mu - \sigma^2/2 \)

  This will be explained later in the course.

What does this mean? (2)

- The variance of the terminal stock price \( S_T \) increases with time and variance:
  \[ \sigma = S_0 \exp \left( \mu + \frac{\sigma^2}{2} \right) T \sqrt{T} \exp \left( (\sigma^2) - 1 \right) \]
  \[\sigma^2 = T S_0^2 \left[ \exp \left( (\sigma^2) - 1 \right) \right] \]

Wikipedia

**Arithmetic moments**

If \( X \) is a lognormally distributed variable, its expected value (\( E \)) is the arithmetic:

\[ E[X] = e^{\mu + \frac{1}{2} \sigma^2} \]

\[ \text{Var}[X] = (e^{\sigma^2} - 1) e^{2 \mu + \sigma^2} = (e^{\sigma^2} - 1) (E[X])^2 \]

\[ \text{s. d.}[X] = \sqrt{\text{Var}[X]} = e^{\mu + \frac{1}{2} \sigma^2} \sqrt{e^{\sigma^2} - 1} \]

Comparing \( S_T \) to \( E(S_T) \), 100 simulations

![Comparison of terminal price simulations to theoretical expected value](image)

Simulation of terminal stock price

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Expected ( S_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35.5953</td>
</tr>
<tr>
<td>2</td>
<td>38.2039</td>
</tr>
<tr>
<td>3</td>
<td>36.2609</td>
</tr>
<tr>
<td>4</td>
<td>45.9779</td>
</tr>
<tr>
<td>5</td>
<td>57.2488</td>
</tr>
<tr>
<td>6</td>
<td>40.7216</td>
</tr>
<tr>
<td>7</td>
<td>66.7042</td>
</tr>
</tbody>
</table>

101 simulations of terminal price compared to theoretical expected value:

\[ E[S_T] = S_0 \exp \left( \mu + \frac{\sigma^2}{2} \right) T \]