Financial Modeling, 4th edition
Chapter 16: The binomial pricing model

In this Powerpoint

- One-period binomial model
- Solving for option prices using simultaneous equations
- Pricing options using state prices
- Pricing options using risk neutral prices

Binomial model

- Most common option pricing model after Black-Scholes (Chapter 17)
- Can be adapted to many situations
  - Standard options
  - Dividends
  - American options
  - Path-dependent, exotics (Chapter 23)
- Simple to understand and program

Binomial model and Black-Scholes

- Binomial converges to Black-Scholes
  - Convergence holds for most common stochastic processes (explanation below)
- Binomial model can be used to prove Black-Scholes theorem (beyond this book)

One-period binomial: 2 assets

- Stock with price $S_0$ today and price $S_{up} = S_0 \times U$ in up state
  and $S_{down} = S_0 \times D$ in down state
- U and D are one-plus-return on the stock. For example, $U=1+10\%$, $D=1-3\%$.
- Bond costs 1 today and pays off R one period hence in both up and down states
- U, D, and R all stand for one-plus the return (numerical example next slide)

One-period binomial, numerical example

In this example: $S_0 = 50$, $U$ (up move) = 1.10, $D$ (down move) = 0.97, and $R$ (1+interest) = 1.06.
The call option has exercise price of 50.
Pricing the call option

- Take portfolio composed of A stocks and B bonds
- Find A and B such that period 1 portfolio payoffs = call payoffs
- Meaning: Solve these two simultaneous equations:

\[
\begin{align*}
55.4 + 1.06B &= 5 \\
48.5 + 1.06B &= 0
\end{align*}
\]

(portfolio value in up state = call payoff in up state)
(portfolio value in down state = call payoff in down state)

Solution to equations

\[A = 0.7692, \ B = -35.959\]

Meaning: Buy 0.7692 stocks and borrow -35.959 at the risk-free rate. This portfolio replicates the call option payoffs.

The date 0 cost of this portfolio is

\[0.7692 \times S_0 - 35.959 = 0.7692 \times 50 - 35.959 = 3.2656\]

State prices: Alternative interpretation of binomial model

- Denote by \(q_U\) the price today of $1 in the up state at date 1
- Denote by \(q_D\) the price today of $1 in the down state at date 1
- The two state prices should price all assets including stock and bond:

\[
\begin{align*}
\text{Stock: } 55q_U + 48.5q_D &= 50 \\
\text{Bond: } 1.06q_U + 1.06q_D &= 1
\end{align*}
\]

State prices depend only on \(U, D, R\)
Not on stock price \(S_0\)

Solving for \(q_U\) and \(q_D\) using stock and bond prices

\[
\begin{align*}
50 &= 55q_U + 48.5q_D \\
1 &= 1.06q_U + 1.06q_D \\
q_U &= \frac{R - D}{R*(U - D)} = \frac{1.06 - 0.97}{1.06*(1.10 - 0.97)} = 0.6531 \\
q_D &= \frac{U - R}{R*(U - D)} = \frac{1.1 - 1.06}{1.06*(1.10 - 0.97)} = 0.2903
\end{align*}
\]

State prices depend only on \(U, D, R\)
Not on stock price \(S_0\)

\[
\begin{align*}
50 &= 55q_U + 48.5q_D = 50 \times q_U + 50 \times D \times q_D \\
\text{or: } \frac{1}{U} &= 1.10q_U + 0.97q_D \\
1 &= 1.06q_U + 1.06q_D \\
q_U &= \frac{R - D}{R*(U - D)} = \frac{1.06 - 0.97}{1.06*(1.10 - 0.97)} = 0.6531 \\
q_D &= \frac{U - R}{R*(U - D)} = \frac{1.1 - 1.06}{1.06*(1.10 - 0.97)} = 0.2903
\end{align*}
\]
Three-date model (more later)

Since U, D, R are the same at each date, the state prices qU and qD are the same at Date 0 and at Date 1.

Using state prices to price the call and put

Risk-neutral prices in our previous example

Risk-neutral prices are not the actual state probabilities

What's the big deal?

In Excel (note check in rows 11 and 12)

Using the state prices gives the same option prices (rows 13/14)
A digression into pure financial economics

- In general it is **wrong** to price assets at the discounted expected value.
- Example: Interest rate = 10%
  - What is the value of $100 one period hence?
  - What is the value of an asset that pays either $150 or $50 next period with equal probability 0.5
  - Why? This asset is riskier than $100 with certainty.

Risk-neutral pricing

- If we adjust the probabilities from the real probabilities to the risk-neutral probabilities,
- Then we can price assets using the discounted expected value: The expected value discounted by $1+r$.
- THIS IS RISK-NEUTRAL PRICING.

Three-date binomial model

Three-date model: More efficient computation

Extension to multiple periods

5 date model

10.4360 = Call value at Date 0
= PV of terminal option payoffs using state prices on each path
- $q_U^0 * (1.2825 + 4 * q_U^0 * q_D^0)*23.205 + 6 * q_U^0 * q_D^0 * 6.9245$
- $+ 4 * q_U^0 * q_D^0 * 0.1970 + q_D^0 * 0$
General formula: Binomial call valuation

\[
\text{Call price} = \sum_{i=0}^{n} q_i^n \max \left( S \cdot U^i \cdot D^{n-i} - X, 0 \right) \\
\text{Put price} = \sum_{i=0}^{n} q_i^n \max \left( X - S \cdot U^i \cdot D^{n-i}, 0 \right)
\]

Direct pricing

\[
\text{Call price} = \frac{X}{D^n} - S
\]

Using put-call parity

\[
R \cdot \text{Call price} - \text{Put price} = X - S
\]

Easy to implement in VBA (next slide)

VBA function Binomial_eur_call

Function Binomial_eur_call(Up, Down, Interest, Stock, Exercise, Periods)
q_up = (Interest - Down) / (Interest * (Up - Down))
q_down = 1 / Interest - q_up
Binomial_eur_call = 0
For Index = 0 To Periods
Binomial_eur_call = Binomial_eur_call + Application.Combin(Periods, Index) * q_up ^ Index * q_down ^ (Periods - Index) * Application.Max(Stock * Up ^ Index * Down ^ (Periods - Index) - Exercise, 0)
Next Index
End Function

Binomial_eur_call is on FM4 book files

Using Binomial_eur_call and Binomial_eur_put in Excel

American put option

Consider an American at-the-money (X=50) put:
- At each date, put price is either:
  - Present value of future put payoffs
  - Or value of immediate exercise
- Example: at bottom state of Date 1
  - Present value = q_U \cdot 0 + q_D \cdot 2.9550 = 0.8578
  - Immediate exercise = 50 - 48.50 = 1.5
  - Thus an American put would be exercised immediately at this state
- Continuing this example: at Date 0
  - Present value of future payoffs = q_U \cdot 0 + q_D \cdot 1.5
  - Immediate exercise at Date 0 = 0, since put is at-the-money

American put vs European put

American call vs European call

American put is worth 0.4354; European put is worth 0.2490

No extra value to American call! We knew this from Proposition 2 of Chapter 15: It is not optimal to early-exercise an American call on a non-dividend paying stock.
VBA functions in book for American puts and calls

<table>
<thead>
<tr>
<th>Binomial price vs Black-Scholes</th>
<th>American call</th>
<th>European call</th>
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</thead>
<tbody>
<tr>
<td>5.6853</td>
<td>4.7255</td>
<td>5.8400</td>
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</tr>
<tr>
<td>5.8238</td>
<td>4.9213</td>
<td>5.8400</td>
</tr>
</tbody>
</table>

Black-Scholes call 12.8226 = BSCall(B2,B3,B4,B5,B6)

Interest rate, R 1.0020 = EXP(B5*B9)

Down, D 0.9537 = EXP(-B6*SQRT(B9))

Up, U 1.0486 = EXP(B6*SQRT(B9))

n, number of subdivisions of T 25

Sigma 30% Riskiness of stock

r, Annual interest rate 8%

T, Time to option exercise (in years) 0.5000

X, Option exercise price 50.00

S, Current stock price 60

Convergence of binomial to Black-Scholes

- Use an approximation to the lognormal
  \[ \Delta = \frac{T}{n} \]
  \[ R = e^{\mu \Delta} \]
  \[ U = 1 + up = e^{\sigma \Delta} \]
  \[ D = 1 + down = e^{-\sigma \Delta} \]

- As \( \Delta \to 0 \), this converges to the lognormal

Employee stock options

- Call option given to employees as remuneration
- Typical conditions:
  - Vesting period: Option cannot be exercised for a number of years.
  - Option is relinquished by employee if he/she leaves employment before vesting
Hull-White model

- Based on standard binomial call model for stock with dividend
  - Dividend is a percentage of stock price
- Assumes that a percentage $e$ of employees leave firm annually before option vesting (losing their ESOs)
- Assumes that after vesting employees early-exercise first time stock price $S_t$ is a multiple $m$ of option exercise $X$: $S_t \geq mX$

Hull-White model for ESOs

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A BINOMIAL EMPLOYEE STOCK OPTION PRICING MODEL Based on Hull-White (2004)</td>
<td></td>
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<tr>
<td>2</td>
<td>Stock price</td>
<td>SB</td>
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<tr>
<td>3</td>
<td>50</td>
<td>ES0</td>
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<td>4</td>
<td>Option exercise price</td>
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<td>6</td>
<td>Present interest rate</td>
<td>R</td>
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<td>7</td>
<td>5.00%</td>
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<td>8</td>
<td>Riskiness of stock</td>
<td>Sigma</td>
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<tr>
<td>9</td>
<td>35%</td>
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<tr>
<td>10</td>
<td>Stock dividend rate</td>
<td>D</td>
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<td>11</td>
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<tr>
<td>12</td>
<td>Exit rate, $e$</td>
<td>e</td>
</tr>
<tr>
<td>13</td>
<td>10.00%</td>
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<tr>
<td>14</td>
<td>Option exercise multiple, $m$</td>
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</tr>
<tr>
<td>15</td>
<td>3.00</td>
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</tr>
<tr>
<td>16</td>
<td>Number of subdivisions of one year</td>
<td>n</td>
</tr>
<tr>
<td>17</td>
<td>50</td>
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<tr>
<td>18</td>
<td>Employee stock option value</td>
<td>ESO</td>
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<td>Black-Scholes call</td>
<td>BSCall</td>
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<tr>
<td>21</td>
<td>19.18</td>
<td></td>
</tr>
</tbody>
</table>

ESO is a VBA function

ESO is part of the spreadsheet for Chapter 16. It can be copy/pasted as described in the document “Adding Getformula to your spreadsheet.doc”. This document is on the disk to Financial Modeling.